Performance Participation Strategies: OBPP versus CPPP

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Abstract

The goal of this paper is to provide and examine an important extension of the usual portfolio insurance, namely to study the notion of portfolio performance participation. In this framework, the portfolio is based on two risky assets: the first one corresponds to a reserve asset, while the second one is considered as an active asset which has usually both a higher mean and a higher variance. We aim at insuring a given percentage of the reserve asset return, whatever the market fluctuations. The two main performance participation methods are the Option-Based Performance Participation (OBPP) and the Constant Proportion Performance Participation (CPPP). We compare these two portfolio strategies by means of various criteria such as their payoffs at maturity, their four first moments and their cumulative distributions functions. We also compare their dynamic hedging properties by computing in particular their deltas and vegas.

JEL classification : G11, G12, G13.

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1 Introduction

The purpose of portfolio insurance is to allow the investor to limit the loss risk while benefiting in part from a possible rise in the reference financial market. At maturity, the investor recovers at least a given percentage of his initial investment, especially in bearish markets. (see e.g. Aftalion and Portait, 1988; Charlety-Lepers and Portait, 1997; Poncet and Portait, 1997). Two standard portfolio insurance methods are the *Option Based Portfolio Insurance* (OBPI) and the *Constant Proportion Portfolio Insurance* (CPPI). The OBPI has been introduced by Leland and Rubinstein (1976). This portfolio strategy is based on an investment in a risky asset S (usually a financial index such as the S&P 500) covered by a put option written on it. Thus, at maturity, the portfolio value is always higher than the strike of the put. The CPPI has been introduced by Perold (1986) (see also Black and Jones, 1987; Perold and Sharpe, 1988; Black and Perold, 1992). This strategy allocates assets dynamically over time as follows: the investor begins by setting a floor equal to the lowest acceptable portfolio value. The difference between the portfolio value and the floor is called the cushion. Then, he allocates to the risky asset an amount ("the exposure") equal to the cushion multiplied by a predetermined multiple. The remaining funds are invested in the reserve asset, usually T-bills. The key parameter is the multiple. Indeed, the higher the multiple, the higher the portfolio return when the financial market is bullish. However, the higher the multiple, the higher the gap risk (i.e. the portfolio becomes smaller than the floor).

The goal of this paper is to extend portfolio insurance by substituting the fixed capital guarantee[1] for a participation in the performance of a risky asset viewed as a reserve asset. The performance participation constraint is to ensure that the portfolio return will be higher than a given percentage of the reserve asset return, whatever the future financial market fluctuations. Note that the portfolio value guaranteed at maturity is no longer deterministic and can be correlated to the active asset itself. Such structured product can be introduced instead of usual portfolio insurance to overcome problems due for example to too low interest rates. For example, the reserve asset is no longer a money market account but rather a long-term government bond while the active asset is an equity index such as the S&P 500. Black and Perold (1992) have introduced such model (that they still call CPPI), showing mainly that, in the absence of transaction costs, this portfolio strategy is equivalent to investing in perpetual American call options, and that it is optimal for a piecewise-HARA utility function with a minimum consumption constraint. However, the interest of such a methodology is also to be able to invest in significantly risky assets (emerging markets, hedge funds...) while ensuring a minimum return relative to a more standard financial index.

In this paper, first we set up the two main performance participation methods, namely the Option-Based Performance Participation (OBPP) and the Constant Proportion Performance Participation (CPPP). We introduce a quite general model allowing to take account simultaneously of both the

[1] In France, several financial structured products have such implicit "capital guarantee". However, the "*Autorité des Marchés Financiers*" (AMF) does no longer consider them as portfolio insurance but rather prefers to label them "fonds à formule" as those which have not such property.

performance participation condition and the initial budget constraint.[2] We show that they can be both expressed in terms of respectively the Option-Based Portfolio Insurance (OBPI) and the Constant Proportion Portfolio Insurance (CPPI) by choosing the reserve asset as numeraire.[3] We show also that the ratio of the two risky assets play a key role when analyzing the two portfolio strategies and especially its volatility.[4] For the study of the OBPP portfolio, we use results of Margrabe (1978) about the valuation of exchange options. Second, as in Bertrand and Prigent (2005, 2011) for the OBPI and CPPI strategies, we examine and compare OBPI and CPPI strategies by using various criteria. In a first part, we analyze and compare their properties at the end of the investment period by examining their payoffs at maturity, their four first moments and their cumulative distributions functions (cdf).[5] As a by-product, we provide the explicit CPPP multiple value such that the two portfolios have the same expected return. For comparison criteria based on cdfs, we show that none of the two strategies stochastically dominates the other one at the first order. We also examine the cdf of the portfolio values ratio. Finally, in a second part, we compare their dynamic properties. First, we prove that the OBPP method can be viewed as a generalized CPPP where the multiple can evolve stochastically over time. Second, we examine their hedging properties by computing in particular their deltas and vegas. For this latter Greek, as for the OBPI and CPPI strategies, the OBPP and CPPP differ very significantly since their signs are opposite: the OBPP value is increasing with respect to the volatility of the ratio of the two risky assets whereas, for the CPPP, it is the converse.

The paper is organized as follows. Section 2 sets out the two strategies. Section 3 compares the two strategies at maturity by means of their payoffs, their four first moments and their cdfs. Section 4 is devoted to dynamic and hedging properties. Finally, section 5 concludes.[6]

[2] See Kraus et al (2010) for the study of a similar model but without taking explicitly the budget constraint into account.

[3] See Black and Perold (1992) for a similar result. See also Bajeux-Besnamou and Portait (1997) for another illustration of the use of a numeraire.

[4] We consider the Geometric Brownian framework in order to get explicit formulas and to prove main properties. Of course, more general stochastic processes can be introduced but they require the use of Monte Carlo simulations.

[5] See Zagst et al (2019) for the comparison of the option-based performance participation with the usual option-based portfolio insurance.

[6] Most of the proofs are gathered in the Appendix.

2 The Performance Participation Strategies

In what follows, we begin by setting up both OBPP and CPPP portfolio values. Then, to assess these two strategies, we begin by comparing their payoffs at maturity. Thereafter, to better take account of their respective probability distributions, we compare their first four moments and their cumulative distribution functions (cdf). Finally, we examine their dynamic properties, especially the computation of their Greeks for hedging purpose.

The portfolio manager is assumed to invest in two financial risky assets denoted by S_1 and S_2. We assume that they are diffusion processes which are solutions of:

$$\begin{aligned} dS_{1,t} &= S_{1,t}\left[\mu_{S_1}dt + \sigma_{S_1}dW_{1,t}\right], \\ dS_{2,t} &= S_{2,t}\left[\mu_{S_2}dt + \sigma_{S_2}\rho dW_{1,t} + \sigma_{S_2}\sqrt{1-\rho^2}dW_{2,t}\right], \end{aligned} \quad (1)$$

where $W_t = (W_{1,t}, W_{2,t})_t$ is a standard two-dimensional Brownian motion with respect to its own filtration $(\mathcal{F}_t)_t$. We assume that the volatilities σ_{S_1} of asset S_1 and σ_{S_2} of asset S_2 are strictly positive and that their instantaneous correlation ρ is not null. The period of time considered is $[0, T]$. The strategies are self-financing. The portfolio manager aims at providing a predetermined participation of a reserve asset (here the risky asset S_1) while investing in an active asset (here the risky asset S_2). Note that usually the active asset S_2 (typically an equity index) is riskier than the reserve asset S_1 (a government bond for example) while it provides a higher instantaneous expected return yielding to conditions $\mu_{S_2} \geq \mu_{S_1}$and $\sigma_{S_2} \geq \sigma_{S_1}$. The predetermined participation of the reserve asset S_1 is to guaranty that portfolio return V_T/V_0 is higher than a given proportion b of asset return $S_{1,T}/S_{1,0}$ (with $0 < b < 1$).

2.1 The Option-Based Performance Participation (OBPP)

The Option-Based Performance Participation strategy aims at providing a predetermined participation of a reserve asset (here the risky asset S_1) by using an adequate exchange option between the active asset S_2 and given a shares of asset S_1, namely the option with payoff $Max(S_{2,T}, aS_{1,T})$ which is the best of the two assets aS_1 and S_2. The parameter a denotes the performance participation on asset S_1 in this exchange option whose payoff can also be expressed as $S_{2,T} + (aS_{1,T} - S_{2,T})^+$. This latter formula shows that this exchange option corresponds to the purchase of asset S_2 covered by a put written on it with "strike" $aS_{1,T}$. Equivalently, we get also the payoff $aS_{1,T} + (S_{2,T} - aS_{1,T})^+$. Note that usually the active asset S_2 is riskier than the reserve asset S_1 while it provides a higher instantaneous expected return yielding to conditions $\mu_{S_2} \geq \mu_{S_1}$and $\sigma_{S_2} \geq \sigma_{S_1}$. The OBPP portfolio is based on this exchange option. To adjust the value of the portfolio, we adequately[7] duplicate this basic structured product by purchasing q shares of it . Thus, the portfolio value V^{OBPP} is given at the terminal date by:

$$V_T^{OBPP} = q\left[aS_{1,T} + (S_{2,T} - aS_{1,T})^+\right]. \quad (2)$$

[7] See subsection 2.1.2.

This relation shows that the insured amount at maturity corresponds to $(qa)\, S_{1,T}$.

Remark 1 *The performance participation is defined as follows: the portfolio return must be always higher than a predetermined fraction b of the reserve asset return. We must have:*

$$\left(V_T^{OBPP}/V_0\right) \geq b\left(S_{1,T}/S_{1,0}\right), \text{ with } 0 < b < 1. \tag{3}$$

Since at maturity we want to recover exactly $b\left(S_{1,T}/S_{1,0}\right)$ if the call is not exercisable (i.e. $S_{2,T} < aS_{1,T}$), the coefficient qa is set equal to $bV_0/S_{1,0}$.

Denote by $V^{exc}\left(t, T, S_2, aS_1\right)$ the value at time t of the exchange option between S_2 and aS_1. Then the value V_t^{OBPP} of the OBPP portfolio at any time 0 is given by:

$$V_t^{OBPP} = qaS_{1,t} + qV^{exc}\left(t, T, S_2, aS_1\right). \tag{4}$$

2.1.1 Valuation of the OBPP strategy

In what follows, we show how to valuate the OBPP strategy.

Remark 2 *The value of the exchange option can be expressed in terms of asset S_1 considered as numeraire. Indeed, $V^{exc}\left(t, T, S_2, aS_1\right)/S_{1,t}$ is equal to the value of the call option with underlying S=S_2/S_1, strike a, zero interest rate and volatility $\widetilde{\sigma}$ which is the volatility of the process S equal to the ratio S_2/S_1 (i.e. the value of asset S_2 expressed in terms of asset S_1 as numeraire) given by:*

$$\widetilde{\sigma} = \sqrt{\sigma_{S_1}^2 + \sigma_{S_2}^2 - 2\rho\sigma_{S_1}\sigma_{S_2}}. \tag{5}$$

We have:

$$V^{exc}\left(t, T, S_2, aS_1\right)/S_1 = Call\left(t, T, S_2/S_1, a, r = 0, \widetilde{\sigma}\right). \tag{6}$$

Therefore, at any time $t \in [0, T]$, the OBPP strategy is equivalent to the purchase of $qS_{1,t}$ shares of the OBPI strategy based on the ratio $S = S_2/S_1$ as underlying risky asset with initial investment value $V_0/S_{1,0}$.[8] *Indeed, using asset S_1 as numeraire, we get the following relation:*

$$\frac{V_T^{OBPP}}{S_{1,T}} = q\left[a + \left(S_{2,T}/S_{1,T} - a\right)^+\right] = qV^{OBPI}\left(T, T, S = S_2/S_1, V_0/S_{1,0}, a, r = 0, \widetilde{\sigma}\right). \tag{7}$$

Previous result implies that, for any time $t \in [0, T]$, we get:

$$\frac{V_t^{OBPP}}{S_{1,t}} = qV^{OBPI}\left(t, T, S = S_2/S_1, V_0/S_{1,0}, a, r = 0, \widetilde{\sigma}\right). \tag{8}$$

[8] Recall that the OBPI strategy satisfies:

$$V_T^{OBPI} = q\left[K + \left(S_T - K\right)^+\right],$$

where qK is set equal to a fixed proportion of the initial invested amount V_0, namely $qK = pV_0$ (see e.g. Bertrand and Prigent, 2005). We denote its current value by $V^{OBPI}\left(t, T, S, V_0, K, r, \sigma\right)$ where r is the risk-free interest rate and σ is the volatility of the risky asset S.

In the Black and Scholes framework, recall that the value of $Call\left(t,T,S_2/S_1,a,r=0,\widetilde{\sigma}\right)$ is equal to:

$$Call\left(t,T,S_2/S_1,a,r=0,\widetilde{\sigma}\right)=\left(S_{2,t}/S_{1,t}\right)N\left(d_{1,t}\right)-aN\left(d_{2,t}\right),$$

where N denotes the cumulative distribution function (cdf) of the standard univariate Gaussian distribution and

$$\begin{cases} d_{1,t}=\dfrac{Log\left[\frac{S_{2,t}}{aS_{1,t}}\right]+\frac{1}{2}\widetilde{\sigma}^2(T-t)}{\widetilde{\sigma}\sqrt{(T-t)}} \\ d_{2,t}=d_{1,t}-\widetilde{\sigma}\sqrt{(T-t)} \end{cases}.$$

We deduce (see Margrabe, 1978):

$$V^{exc}\left(t,T,S_2,aS_1\right)=S_{2,t}N\left(d_{1,t}\right)-aS_{1,t}N\left(d_{2,t}\right). \tag{9}$$

2.1.2 Determination of the number of shares q and of the performance participation a

To take account of both participation and budget constraint $V_0^{OBPP}=V_0$, we have to solve the following system of equations:

$$\begin{cases} qa=bV_0/S_{1,0} \\ V_0=q\left[aS_{1,0}+V^{exc}\left(0,T,S_2,aS_1\right)\right] \end{cases},$$

which implies that a must satisfy:

$$\frac{1-b}{b}=\frac{V^{exc}\left(0,T,S_2,aS_1\right)/S_{1,0}}{a}=\frac{Call\left(t,T,S_2/S_1,a,r=0,\widetilde{\sigma}\right)}{a}. \tag{10}$$

Since $\varphi(b)=\frac{1-b}{b}$ is strictly decreasing and bijective from $]0,1[$ to $]0,+\infty[$ and $\psi(a)=\frac{Call(t,T,S_2/S_1,a,r=0,\widetilde{\sigma})}{a}$ is strictly decreasing and bijective from $]0,+\infty[$ to $]0,+\infty[$, there exists one and only one solution a^*. Then, we set $q^*=\frac{bV_0/S_{1,0}}{a^*}$.

2.2 The Constant Proportion Performance Participation (CPPP)

The CPPP method consists of managing a dynamic portfolio so that its value V_t^{CPPP} is above the floor $F_t=\delta S_{1,t}$ at any time t, where δ is fixed. The difference $V_t^{CPPP}-F_t$ at any time t in $[0,T]$ is called the cushion, denoted by C_t.[9] Denote by e_t the exposure, which is the total amount invested in the risky asset S_2. The CPPP method consists of letting $e_t=mC_t$ where m is a constant parameter called the multiple. The interesting case is when $m>1$, that is, when the payoff function is convex with respect to the asset S_2 payoff. The value of portfolio V_t^{CPPP} at any time t in the period $[0,T]$ is given by[10]:

$$V_t^{CPPP}=\delta S_{1,t}+\alpha_t.S_{1,t}^{1-m}S_{2,t}^{m},$$

[9] To insure that the cushion is always positive, we can set $C_t=Max\left(V_t^{CPPP}-\delta S_{1,t},0\right)$. It implies that, if the portfolio value becomes smaller than the floor, then the whole portfolio value is invested in asset S_1.

[10] Details about this formula are provided in the Appendix.

where

$$\alpha_t = \left(\frac{V_0 - \delta S_{1,0}}{S_{1,0}^{1-m} S_{2,0}^m} \right) \exp[\beta t] \text{ and } \beta = -\frac{1}{2} m(m-1)\widetilde{\sigma}^2.$$

Thus, the CPPP method is parametrized by δ and m.

Remark 3 *Note that we have:*

$$\left(\frac{V_0 - \delta S_{1,0}}{S_{1,0}^{1-m} S_{2,0}^m} \right) = \left(\frac{V_0/S_{1,0} - \delta}{(S_{2,0}/S_{1,0})^m} \right).$$

Additionally, $\widetilde{\sigma}$ is the volatility of the process S_2/S_1. Therefore, as for the OBPP case, by using asset S_1 as numeraire, the CPPP portfolio value can be expressed in terms of reserve asset S_1 as follows:[11]

$$\begin{aligned} \frac{V_t^{CPPP}}{S_{1,t}} &= \delta + \alpha_t . (S_{2,t}/S_{1,t})^m \\ &= V_t^{CPPI}(t, T, S = S_2/S_1, V_0/S_{1,0}, \delta, r = 0). \end{aligned}$$

3 Comparison between OBPP and CPPP at maturity

3.1 Comparison of the payoff functions

The OBPP has just one parameter, the participation coefficient a. In order to compare the two methods, first the initial amounts V_0^{OBPI} and V_0^{CPPI} are assumed to be equal to the same initial invested amount V_0. Secondly, the two strategies are assumed to provide the same participation coefficient a, which implies to choose the performance participation δ of the CPPP strategy equal to qa. Hence, $F_T = qaS_{1,T}$ and then $F_0 = qaS_{1,0}$. Note that these two conditions do not impose any constraint on the multiple, m. Therefore, we can consider several CPPP strategies for various values of the multiple m. [12]Note that, due to the absence of arbitrage and since $V_0^{OBPI} = V_0^{CPPI} = V_0$, neither of the two payoffs is higher than the other for all terminal values of the risky asset. The two payoff functions intersect one another. We choose the following numerical example with typical values for a financial market with a long term government bond as reserve asset and an equity index as active asset:

$$S_{0,1} = S_{0,2} = 100, \mu_{S_1} = 5\%, \sigma_{S_1} = 6\%, \mu_{S_2} = 9\%, \sigma_{S_2} = 20\%, \rho = -0.15, T = 1 \text{ year}, b = 0.9. \quad (11)$$

[11] Recall that the CPPI strategy associated to the multiple m satisfies:

$$V_T^{CPPI} = pV_0 + \alpha_T^{CPPI} S_T^m,$$

where p is a guaranteed proportion of the initial invested amount V_0 (see e.g. Bertrand and Prigent, 2005). The coefficient α_T is given by:

$$\alpha_T^{CPPI} = \frac{\left(V_0 - pV_0 e^{-rT}\right)}{S_0^m} e^{\beta^{CPPI} T} \text{ with } \beta^{CPPI} = -\frac{1}{2} m(m-1)\sigma^2,$$

where r is the risk-free interest rate and σ is the volatility of the risky asset S. We denote its current value by $V^{CPPI}(t, T, S, V_0, p, r, \sigma)$

[12] Note that the multiple must not be too high as shown for example in Prigent (2001) or in Bertrand and Prigent (2005).

Figure 1 illustrates the OBPP payoff as functions of the two assets. We note that the payoff is convex and increasing with respect to the active asset. Figure 2 displays the CPPP payoff as functions

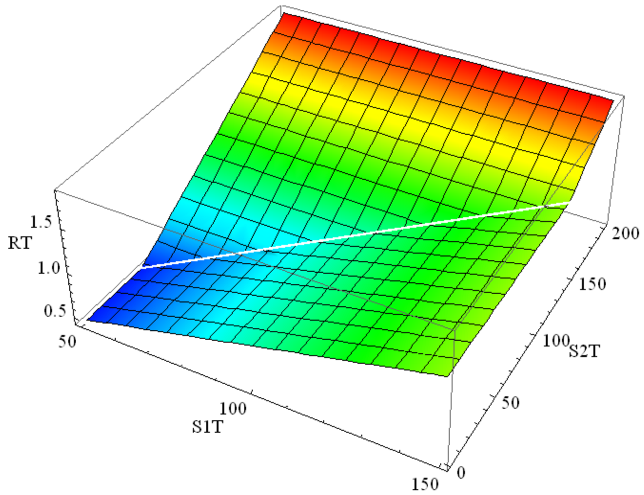

Figure 1: OBPP return as function of the two assets $S_{T,1}$ and $S_{T,2}$

of the two assets. As for the OBPP payoff, the CPPP payoff is convex and increasing with respect to the active asset.

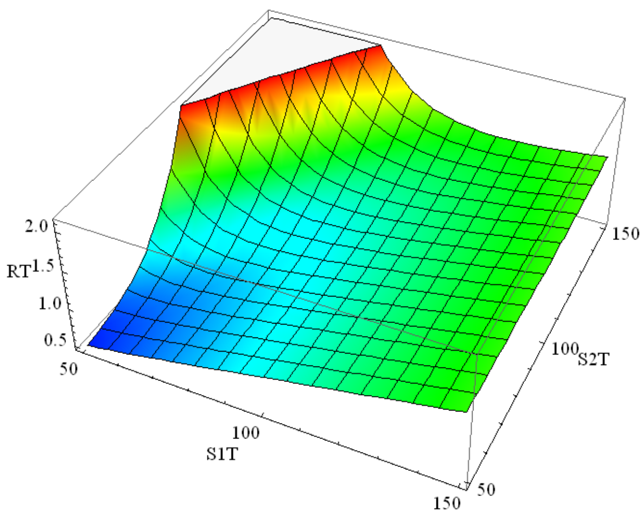

Figure 2: CPPP return as function of the two assets $S_{T,1}$ and $S_{T,2}$

Figure 3 illustrates the comparison of both OBPP and CPPP payoffs as function of the ratio process $S = S_2/S_1$. Note that as m increases, the payoff function of the CPPP becomes more convex with respect to the ratio $S = S_2/S_1$.

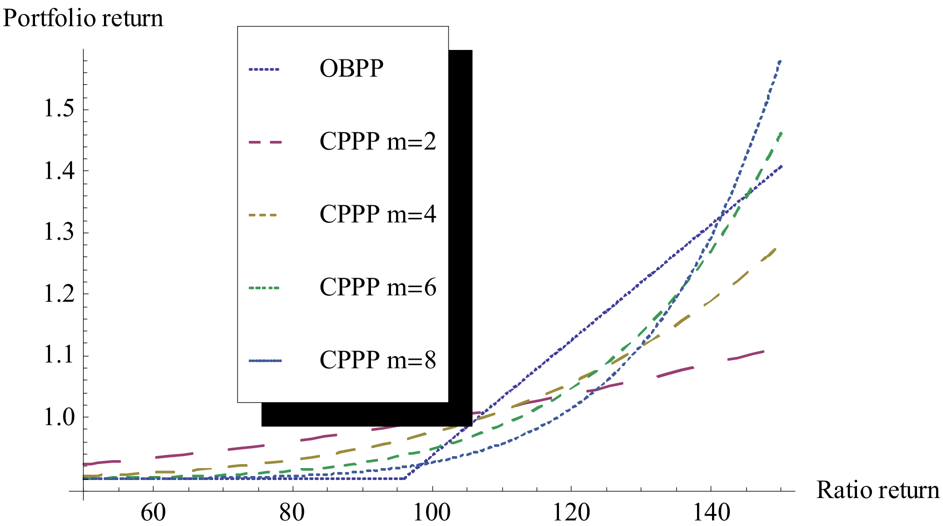

Figure 3: CPPP and OBPP payoffs in terms of the numeraire S_1 as functions of the ratio $S = S_2/S_1$

We can check in this example that the two curves intersect one another for the different values of m considered ($m = 2$, $m = 4$, $m = 6$ and $m = 8$). CPPP performs better for large fluctuations of the ratio S while OBPP performs better around the initial value of ratio S and for its moderate increases.

3.2 Comparison of the expectation, variance, skewness and kurtosis

In what follows, we examine the first four moments of both strategies. Indeed, since their payoffs are not linear with respect to the two assets, the standard mean-variance approach is not sufficiently suitable due to significant skewness and kurtosis. We can consider various values of the multiple m to compare the first four moments. However, it is interesting to examine the special case corresponding to the equality of the expected returns. We determine the multiple corresponding to this additional constraint in the following subsection.

3.2.1 Equality of return expectations

Let us denote the rates of portfolio returns by R_T^{OBPP} and R_T^{CPPP}. Recall that we assume that $\mu_{S_2} > \mu_{S_1}$.

Proposition 4 *For any given participation coefficient a, there exists a unique value $m^*(a)$ of the* *multiple such that $E[R_T^{OBPP}] = E[R_T^{CPPP}]$. In the Black and Scholes framework, it is given by:*

$$m^*(a) = 1 + \left(\frac{1}{(\mu_{S_2} - \mu_{S_1})\,T}\right) \ln\left(\frac{C(0, T, S_2/S_1, a, r = \mu_{S_2} - \mu_{S_1}, \widetilde{\sigma})}{C(0, T, S_2/S_1, a, r = 0, \widetilde{\sigma})}\right), \tag{12}$$

where $C(0, T, S, a, r, \sigma)$ denotes the Black-Scholes value of the call with underlying S, strike a , interest rate r and volatility σ, evaluated at time 0 and with investment horizon T.

Note that, from relation 10, we deduce that coefficient a is increasing with respect to performance participation percentage b. Then, from relation 12, we deduce that the multiple $m^*(a)$ is an increasing function of the performance participation coefficient a. Therefore, m^* is also increasing with respect to b, as shown in Figure 4.

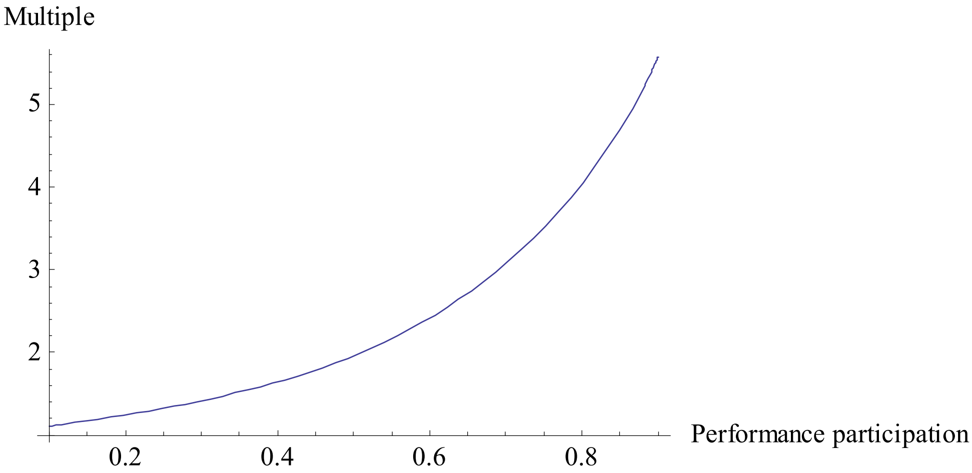

Figure 4: Multiple of the CPPP strategy such that its mean is equal to that of the OBPP. Illustration as function of the performance participation percentage b

Note that both the expected returns of the OBPP and the CPPP strategies are decreasing with respect to coefficient performance participation percentage b since both the number of shares q and the call values $C(0, T, S_2/S_1, a, r = \mu_{S_2} - \mu_{S_1}, \widetilde{\sigma})$ and $C(0, T, S_2/S_1, a, r = \mu_{S_2} - \mu_{S_1}, \widetilde{\sigma})$ are decreasing with respect to b. From proposition 4, as for the comparison of standard OBPI and CPPI, we can deduce that there exists at least one value for m, for example the value m^*, such that the OBPP strategy dominates in a mean-variance sense the CPPI.

3.2.2 Numerical comparison of the first four moments

Using parameter values of the numerical base case given in relation 11, we provide the numerical values of the first four moments for the equality of expected returns case in Table 1. We consider three values of the performance participation percentage b, namely $b = 0.5, 0.7$ and 0.9.

Table 1

Comparison of the first four moments

	$b = 0.5$	$m^* = 4.6$	$b = 0.7$	$m^* = 3.07$	$b = 0.9$	$m^* = 5.57$
	OBPP	CPPP	OBPP	CPPP	OBPP	CPPP
expectation	9.41%	9.41%	9.25%	9.25%	7.552%	7.552%
volatility	22.09%	24%	21.56%	25.63%	15.94%	23.07%
relative skewness	0.61	10.16	1.49	2.52	1.47	10.21
relative kurtosis	3.67	3.83	5.46	15.98	5.80	332

The OBPP dominates the CPPP in a mean-variance sense. However, the CPPP has a significantly higher positive relative skewness than the OBPP. This is due to its more prominent convexity Hence with respect to this criterion, CPPP should be preferred to OBPP. Looking at relative kurtosis, it is the converse since the CPPP relative kurtosis is much higher than that of OBPP.

Remark 5 *For a given performance participation percentage b, as soon as the multiple m is higher than m^*, both the expected return and variance of the CPPP strategy are higher than those of the OBPP strategy. Therefore, no strategy dominates the other with respect to the mean-variance criterion. If $m < m^*$, for a small difference between m and m^*, the variance of the OBPP strategy remains smaller than that of the CPPP. Consequently, the OBPP strategy strictly dominates the CPPP strategy. For a sufficiently large difference $m^* - m$, no strategy dominates the other one.*

3.3 Comparison of quantiles

Due to the presence of asymmetry and fat tails in the probability distributions, it is more convenient to examine the whole distribution by comparing their cdfs. Figure 5 shows that the OBPP and CPPP cdf curves intersect, implying that there is no stochastic dominance at the first order.

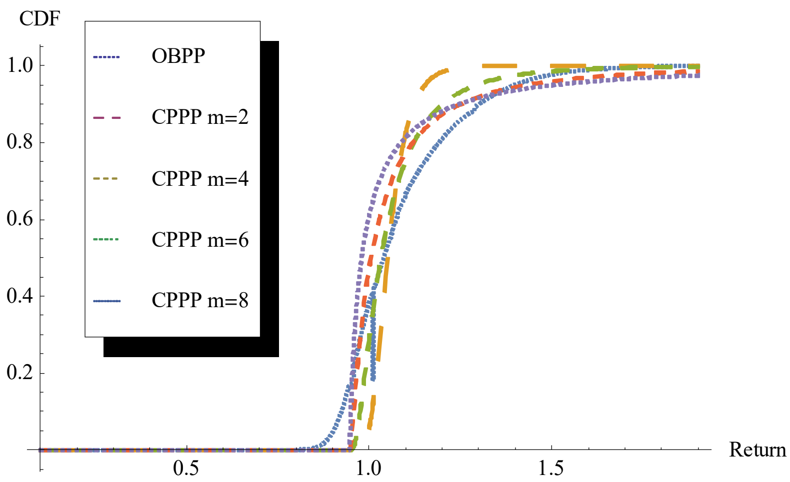

Figure 5: Cumulative distribution functions

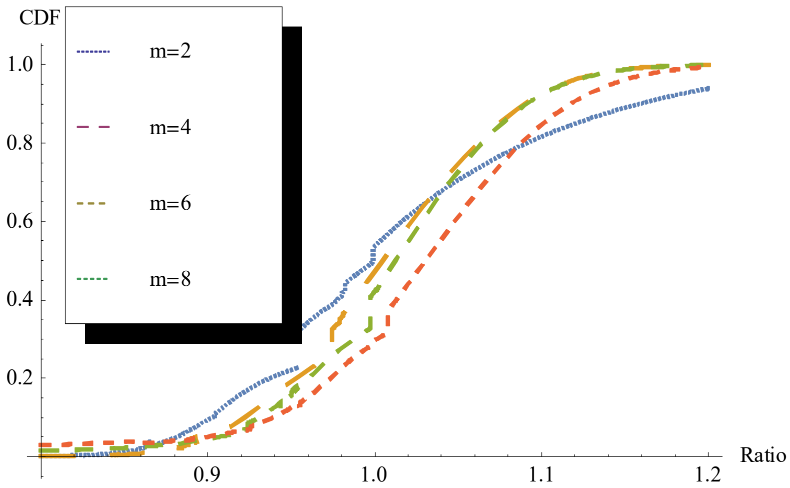

Figure 6: Cumulative distribution function of the ratio V_T^{OBPP}/V_T^{CPPP}

To go further on the comparison of the two payoffs, we investigate the probability distribution of the ratio OBPP value on the CPPP value. Note that using remarks 2 and 3, we deduce:

$$\frac{V_T^{OBPP}}{V_T^{CPPP}} = \frac{V^{OBPI}\left(t, T, r = 0, S = S_2/S_1, V_0/S_{1,0}, a\right)}{V^{CPPI}\left(t, T, r = 0, S = S_2/S_1, V_0/S_{1,0}, a\right)}.$$

Using parameter values of the numerical base case given in relation 11, Figure 6 displays the cumulative distribution function of $\frac{V_T^{OBPP}}{V_T^{CPPP}}$ for different values of the multiple, namely $m = 2$, 4, 6 and 8.

Figure 6 shows that the probability that the OBPP value is smaller than the CPPP value is minimal for the smallest multiple value but is not monotonic with respect to the CPPP multiple. However the higher the multiple, the smaller the probability that the OBPP value is significantly higher than the CPPP value (ratio values higher than 10%).

4 Comparison of the dynamic behaviors of OBPP and CPPP

Even if the OBPP is mainly a buy-and-hold strategy (i.e. at initial date, the investor purchases the financial assets once and for all), its optional component has to be dynamically hedged from the portfolio manager point of view. Additionally, as proved in what follows, the OBPP strategy can be considered as a generalized CPPP by means of a variable multiple which measures its risk exposure. In the Black and Scholes model, there exists a perfect hedging strategy and both performance participation methods can be analyzed and compared dynamically.

4.1 OBPP as a generalized CPPP

For the CPPP method, the multiple is the fundamental parameter since it determines the amount allocated in the active asset at any time. Therefore, from the theoretical point of view, it is interesting to express the OBPP as a generalized CPPP with a corresponding generalized multiple (at least in the Black and Scholes framework).

Proposition 6 *The OBPP strategy is equivalent to an extension of the CPPP one, provided that the multiple satisfies*

$$m^{OBPP}(t, S_t) = \frac{(S_{2,t}/S_{1,t})\, N(d_{1,t})}{Call\,(t, T, S_{2,t}/S_{1,t}, a, r = 0, \widetilde{\sigma})}. \tag{13}$$

Proof. Recall that:

$$V_T^{OBPP} = q\left(aS_{1,T} + (S_{2,T} - aS_{1,T})^+\right)$$

Thus:

$$V_t^{OBPP} = q\left(aS_{1,t} + Call\,(t, T, S_{2,t}/S_{1,t}, a, r = 0, \widetilde{\sigma})\right).$$

Therefore, the OBPP is always above the floor $F_t = qaS_{1,t}$ and $qS_{1,t}Call\,(t, T, S_{2,t}/S_{1,t}, a, r = 0, \widetilde{\sigma})$ corresponds to the cushion $C_t^{OBPP} = \left(V_t^{OBPP} - F_t\right)$ at any time t during the portfolio management period $[0, T]$. By definition, this cushion is equal to $\frac{e_t^{OBPP}}{m_t^{OBPP}}$. Here, the cushion is simply the call and the exposure is the total amount invested in the risky asset, equal to $S_{2,t}\frac{\partial V_t^{OBPP}}{\partial S_{2,t}}$. We note that:

$$\frac{\partial V_t^{OBPP}}{\partial S_{2,t}} = qS_{1,t}\frac{\partial Call\,(t, T, S_{2,t}/S_{1,t}, a, r = 0, \widetilde{\sigma})}{\partial S_{2,t}},$$

which is equal to $qS_{2,t}N(d_{1,t})$. Finally, the desired result for the multiple is obtained:

$$m_t^{OBPP} = \frac{(S_{2,t}/S_{1,t})\, N(d_{1,t})}{Call\,(t, T, S_{2,t}/S_{1,t}, a, r = 0, \widetilde{\sigma})}.$$

■

Therefore, the OBPP multiple is a function of the ratio S of the two asset values[13]. As illustrated by figure 7, this generalized multiple is a decreasing function with respect to the ratio S.

[13] Such more general multiples have been introduced and studied in Prigent (2001).

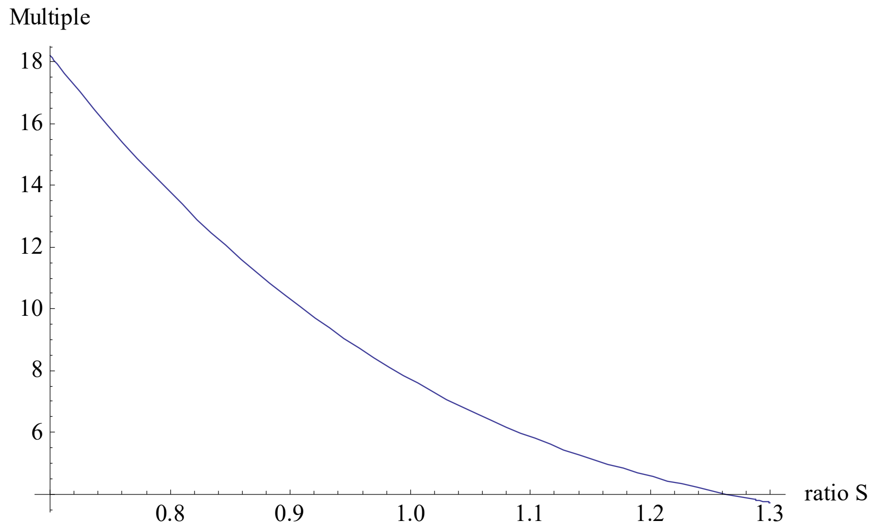

Figure 7: OBBP multiple as function of the ratio of the two assets (at t=T/2).

Recall that the generalized multiple m_t^{OBPP} is the ratio of the exposure on the cushion C_t^{OBPP}. Therefore, it can be very high if either the cushion is too small, or the exposure is too high. In such a case, there exists a significant gap risk if the market suddenly drops.

4.2 The Deltas

The deltas of the OBPP are obviously based on the delta of the call, which yields to:

$$\begin{aligned}\Delta_{1,t}^{OBPP} &= \frac{\partial V_t^{OBPP}}{\partial S_{1,t}} = qa\left[1 - N\left(d_{2,t}\right)\right],\\ \Delta_{2,t}^{OBPP} &= \frac{\partial V_t^{OBPP}}{\partial S_{2,t}} = qN\left(d_{1,t}\right).\end{aligned}$$

For the CPPP, they are given by:

$$\begin{aligned}\Delta_{1,t}^{CPPP} &= \frac{\partial V_t^{CPPP}}{\partial S_{1,t}} = qa + \alpha_t\left(1-m\right)S_{1,t}^{-m}S_{2,t}^{m} = qa + \alpha_t\left(1-m\right)S_t^{m},\\ \Delta_{2,t}^{CPPP} &= \frac{\partial V_t^{CPPP}}{\partial S_{2,t}} = \alpha_t m S_{1,t}^{1-m}S_{2,t}^{m-1} = \alpha_t m S_t^{m-1}.\end{aligned}$$

For the deltas $\Delta_{1,t}$, we note that $\Delta_{1,t}^{OBPP}$ is always positive but smaller than qa which is equal to the performance participation percentage b times the ratio of initial values $V_0/S_{1,0}$. For high values of the ratio process S, and high values of the multiple m, $\Delta_{1,t}^{CPPP}$ can be negative due to the leverage effect on the reserve asset resulting from the strong convexity of CPPP value for high multiples (see Figure 8). For both the OBPP and CPPP strategies, the deltas $\Delta_{2,t}$ are always positive (see similar results for the OBPI and CPPI strategies). Due to the strong convexity of CPPP value, $\Delta_{2,t}^{CPPP}$ can be significantly higher than q, which is the upper bound of $\Delta_{2,t}^{OBPP}$ (see Figure 9).

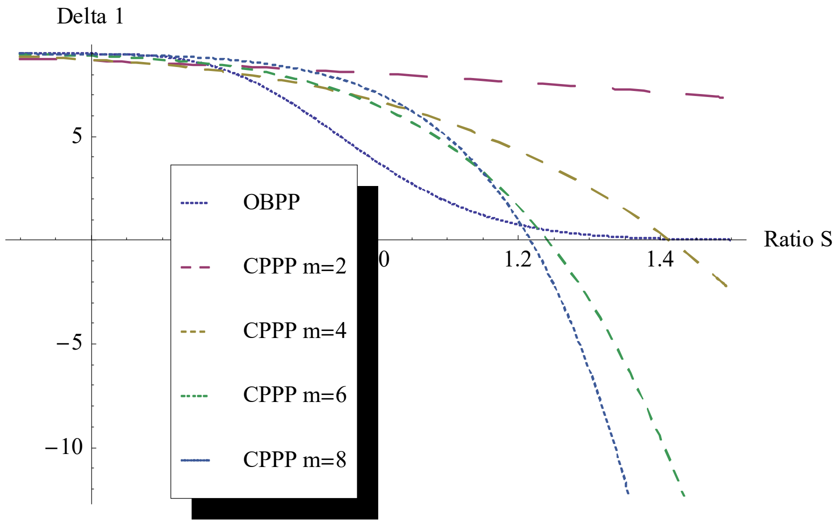

Figure 8: Comparison of Delta 1 as function of the ratio of the two assets S.

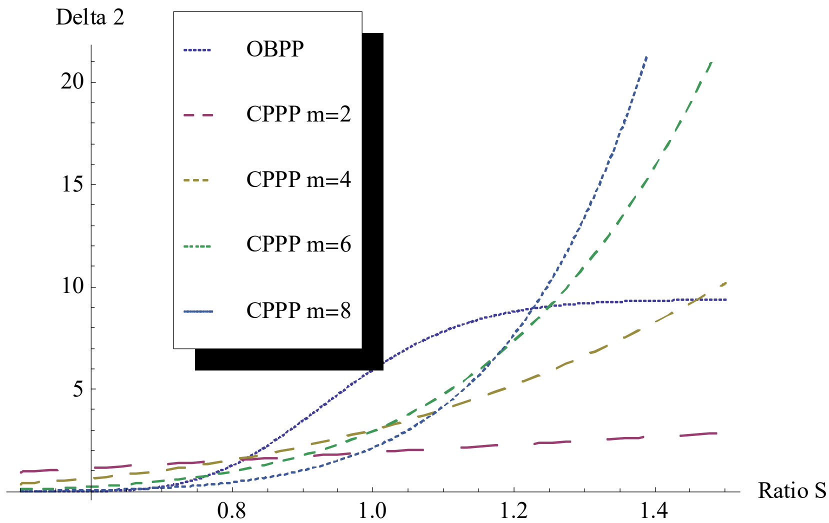

Figure 9: Comparison of Delta 2 as function of the ratio of the two assets.

4.3 The Gammas

The Gammas of the OBPP are given by:

$$\begin{aligned}
\Gamma_{1,t}^{OBPP} &= \frac{\partial^2 V_t^{OBPP}}{\partial S_{1,t}^2} = qa\frac{N'\left(d_{2,t}\right)}{S_{1,t}\widetilde{\sigma}\sqrt{(T-t)}} > 0, \\
\Gamma_{2,t}^{OBPP} &= \frac{\partial^2 V_t^{OBPP}}{\partial S_{2,t}^2} = q\frac{N'\left(d_{1,t}\right)}{S_{2,t}\widetilde{\sigma}\sqrt{(T-t)}} > 0, \\
\widetilde{\Gamma}_{1,2,t}^{OBPP} &= \frac{\partial^2 V_t^{OBPP}}{\partial S_{1,t}\partial S_{2,t}} = -q\frac{N'\left(d_{1,t}\right)}{S_{1,t}\widetilde{\sigma}\sqrt{(T-t)}} < 0.
\end{aligned}$$

The Gammas of the CPPP are equal to:

$$\begin{aligned}
\Gamma_{1,t}^{CPPP} &= \frac{\partial^2 V_t^{CPPP}}{\partial S_{1,t}^2} = \alpha_t m\,(m-1)\, S_{1,t}^{-(m+1)} S_{2,t}^m = \alpha_t m\,(m-1)\, S_{1,t}^{-1} S_t^m > 0, \\
\Gamma_{2,t}^{CPPP} &= \frac{\partial^2 V_t^{CPPP}}{\partial S_{2,t}^2} = \alpha_t m\,(m-1)\, S_{1,t}^{1-m} S_{2,t}^{m-2} = \alpha_t m\,(m-1)\, S_{1,t}^{-1} S_t^{m-2} > 0, \\
\widetilde{\Gamma}_{1,2,t}^{CPPP} &= \frac{\partial^2 V_t^{CPPP}}{\partial S_{1,t}\partial S_{2,t}} = -\alpha_t m\,(m-1)\, S_{1,t}^{-m} S_{2,t}^{m-1} = -\alpha_t m\,(m-1)\, S_{1,t}^{-1} S_t^{m-1} < 0.
\end{aligned}$$

For both performance participation properties, the two derivatives at the first order $\frac{\partial^2 V_t}{\partial S_{1,t}^2}$ and $\frac{\partial^2 V_t}{\partial S_{2,t}^2}$ are positive while the cross-derivatives are negative. Note that both Hessians are null (see Appendix). For the OBPP strategy, the sensitivities $\Gamma_{1,t}^{OBPP}$ and $\Gamma_{2,t}^{OBPP}$ are first increasing then decreasing while, for the CPPP strategy, $\Gamma_{1,t}^{CPPP}$ and $\Gamma_{2,t}^{CPPP}$ are always increasing and can take high values when the ratio process S increases significantly (see Figures 10 and 11). This is still due to the strong convexity of the CPPP value.

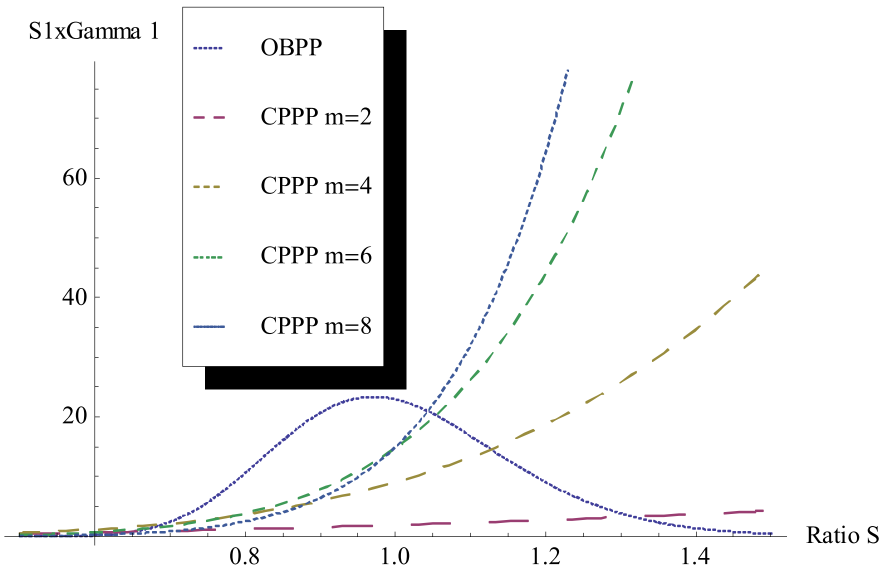

Figure 10: Gamma 1 (times S1) as function of the ratio of the two assets

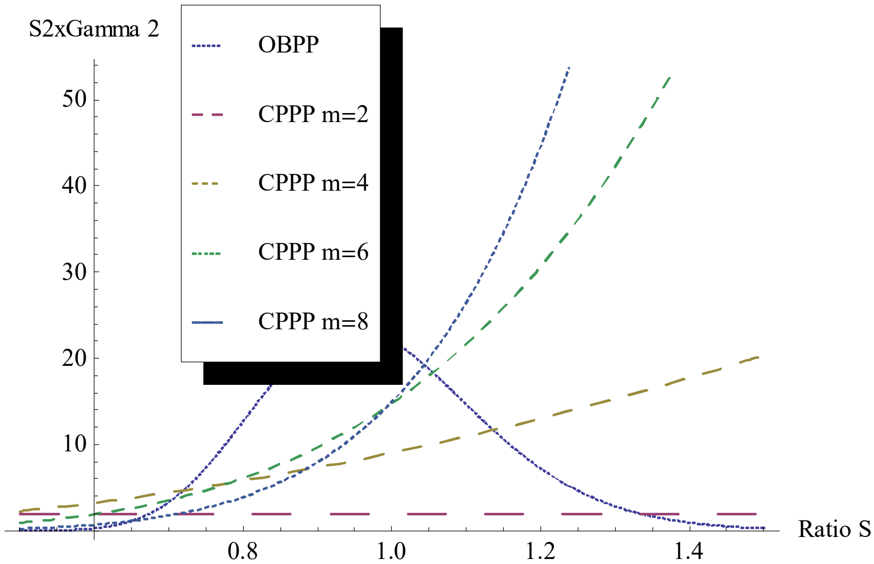

Figure 11: Gamma 2 (times S2) as function of the ratio of the two assets

4.4 The Vegas

In what follows, we consider the sensitivity of the portfolio values to the volatility $\widetilde{\sigma}$ of the ratio process $S = S_2/S_1$.

The Vega of the OBPP is defined as:[14]

$$\mathcal{V}_t^{OBPP} = \frac{\partial V_t^{OBPP}}{\partial \widetilde{\sigma}} = qS_{2,t}\sqrt{(T-t)}N'(d_{1,t}),$$

[14] Recall that the Vega of $Call(t,T,S,K,r,\sigma)$ calculated in the Black and Scholes framework is equal to:

$$\frac{\partial Call(t,T,S,K,r,\sigma)}{\partial \sigma} = S_t\sqrt{(T-t)}N'(d_{1,t}).$$

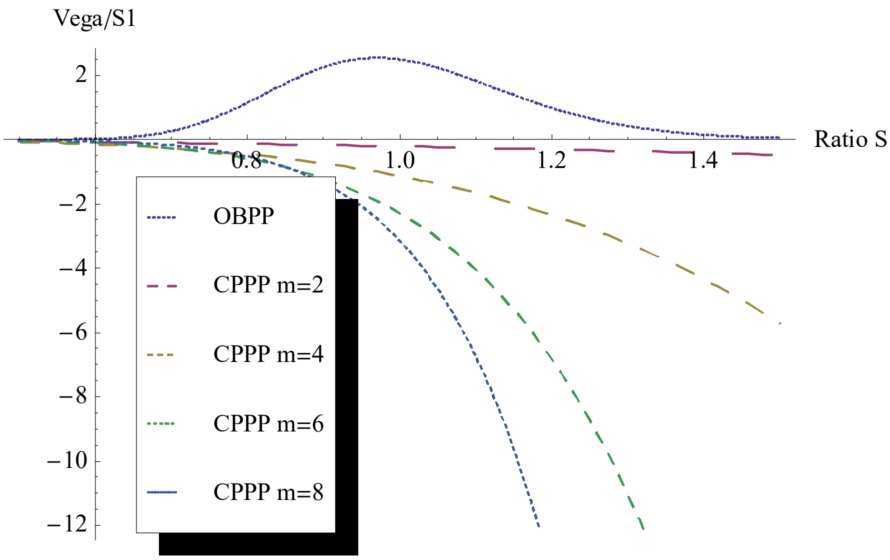

Figure 12: Vega in terms of the reserve asset as function of the ratio of the two assets

For the CPPP, the Vega is given by:[15]

$$\mathcal{V}_t^{CPPP} = \frac{\partial V_t^{CPPP}}{\partial \widetilde{\sigma}} = -m\,(m-1)\,\widetilde{\sigma} t C_t^{CPPP}.$$

Therefore, the sensitivity of the OBPP value with respect to the volatility $\widetilde{\sigma}$ is positive whereas, for the CPPP, it is negative since $m > 1$.

This means that an increase in volatility works against the strategy CPPP. It implies also that, if the ratio of the two assets S_2/S_1 increases moderately while its volatility increases, then the OBPP strategy will be preferred to the CPPP one (see Figure 3). For the strategy CPPP to be the best, the increase in the ratio will have to be relatively large if its volatility increases, for example a large growth of the active asset and/or a significant drop in the price of the reserve asset.

This is similar to the OBPI versus CPPI case: as highlighted by Black and Rouhani (1989), when the volatility increases, the CPPI values decreases. Additionally, the higher the multiple, the more significant the decrease.

[15] Recall that:

$$V_t^{CPPP} = qaS_{1,t} + \alpha_t . S_{1,t}^{1-m} S_{2,t}^m,$$

where

$$\alpha_t = \left(\frac{V_0 - qaS_{1,0}}{S_{1,0}^{1-m} S_{2,0}^m} \right) \exp\left[\beta t\right] \text{ and } \beta = -\frac{1}{2} m(m-1)\widetilde{\sigma}^2.$$

Therefore, we get:

$$\begin{aligned} \frac{\partial V_t^{CPPP}}{\partial \widetilde{\sigma}} &= \left(\frac{V_0 - qaS_{1,0}}{S_{1,0}^{1-m} S_{2,0}^m} \right) \exp\left[\beta t\right] t \frac{\partial \beta}{\partial \widetilde{\sigma}} \\ &= -m(m-1)\widetilde{\sigma} t C_t^{CPPP}. \end{aligned}$$

5 Conclusion

In this paper, we have analyzed and compared the two main performance participation methods, namely the OBPP and CPPP. We have shown that usual comparison criteria such as comparison of their payoffs, of their cdfs with first order stochastic dominance and of their first four moments, do not allow a clear conclusion in favour of one of the two strategies. This is not so surprising since, as proved in the paper, their relative values expressed in terms of the reserve asset as numeraire correspond respectively to OBPI and CPPI strategies the comparison of which leads to similar conclusions. For example, when the ratio of the two processes is around or moderately higher than its initial value, the OBPP value is higher than the CPPP one (at least in the GBM framework). Otherwise, it is the converse. Note also that the OBPP is a buy-and-hold strategy while the CPPP is more flexible but may bear transaction costs and is more sensitive to sudden drops. Further extensions could consider capped OBPP and CPPP, more general dynamics by using explicit results (see e.g. Quittard-Pinon and Randrianarivony (2010) when there are specific jumps) or by using Monte Carlo simulations and also other decision criteria for absolute and relative loss control (see e.g. Mantilla-Garcia, 2014). The optimality of such products could also be examined (see e.g. Bajeux-Besnamou and Portait (1998) for the mean-variance case; Lesne et al. (2001) for a general utility function) and also according to various assumptions about the asset dynamics (see e.g. Bernard and Kwak, 2016). Finally, much flexibility would be allowed for example by introducing a conditional multiple as in Ben Ameur and Prigent (2014).

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6 Appendix

6.1 Determination of the CPPP value

The value at time t of the CPPP portfolio is given by:

$$dV_t^{CPPP} = (V_t^{CPPP} - e_t)\frac{dS_{1,t}}{S_{1,t}} + e_t\frac{dS_{2,t}}{S_{2,t}}.$$

Recall that $V_t^{CPPP} = C_t + F_t$, $e_t = mC_t$ and $F_t = \delta S_{1,t}$. Thus, the cushion value C must satisfy :

$$\begin{aligned} dC_t &= d(V_t^{CPPP} - F_t), \\ &= (V_t^{CPPP} - e_t)\tfrac{dS_{1,t}}{S_{1,t}} + e_t\tfrac{dS_t}{S_t} - dF_t, \\ &= (C_t + F_t - mC_t)\tfrac{dS_{1,t}}{S_{1,t}} + (mC_t)\tfrac{dS_t}{S_t} - dF_t, \\ &= (C_t - mC_t)\tfrac{dS_{1,t}}{S_{1,t}} + (mC_t)\tfrac{dS_{2,t}}{S_{2,t}}. \end{aligned}$$

Therefore, we get:

$$\frac{dC_t}{C_t} = (1-m)\frac{dS_{1,t}}{S_{1,t}} + m\frac{dS_{2,t}}{S_{2,t}},$$

from which we deduce:

$$C_t = C_0 \ \exp\left[\begin{array}{c} \left((1-m)\mu_{S_1} + m\mu_{S2} - 1/2\left[(1-m)^2\sigma_{S_1}^2 + m^2\sigma_{S_2}^2 + 2\rho m(1-m)\sigma_{S_1}\sigma_{S_2}\right]\right)t \\ +(1-m)\sigma_{S_1}W_{1,t} + m\sigma_{S_2}\rho W_{1,t} + m\sigma_{S_2}\sqrt{1-\rho^2}W_{2,t} \end{array}\right],$$

By using the relations:

$$\begin{aligned} S_{1,t} &= S_{1,0}\exp\left[\sigma_{S_1}W_{1,t} + \left(\mu_{S_1} - \frac{1}{2}\sigma_{S_1}^2\right)t\right], \\ S_{2,t} &= S_{2,0}\exp\left[\sigma_{S_2}\rho W_{1,t} + \sigma_{S_2}\sqrt{1-\rho^2}W_{2,t} + \left(\mu_{S_2} - \frac{1}{2}\sigma_{S_2}^2\right)t\right], \end{aligned}$$

we can deduce that:

$$\begin{aligned} \exp\left[\sigma_{S_1}W_{1,t}\right] &= \frac{S_{1,t}}{S_{1,0}}\exp\left[-\left(\mu_{S_1} - \frac{1}{2}\sigma_{S_1}^2\right)t\right], \\ \exp\left[\sigma_{S_2}\sqrt{1-\rho^2}W_{2,t} + \sigma_{S_2}\rho W_{1,t}\right] &= \frac{S_{2,t}}{S_{2,0}}\exp\left[-\left(\mu_{S_2} - \frac{1}{2}\sigma_{S_2}^2\right)t\right]. \end{aligned}$$

By substituting these expressions for $W_{1,t}$ and $W_{2,t}$ into previous expression for C_t, we get:

$$\begin{aligned} C_t &= C_0 \ \exp\left[\left((1-m)\mu_{S_1} + m\mu_{S2} - 1/2\left[(1-m)^2\sigma_{S_1}^2 + m^2\sigma_{S_2}^2 + 2\rho m(1-m)\sigma_{S_1}\sigma_{S_2}\right]\right)t\right] \\ &\quad \times\exp\left[\sigma_{S_1}W_{1,t}\right]^{(1-m)}\exp\left[\sigma_{S_2}\sqrt{1-\rho^2}W_{2,t} + \sigma_{S_2}\rho W_{1,t}\right]^m, \end{aligned}$$

$$\begin{aligned} C_t &= C_0 \ \exp\left[\left((1-m)\mu_{S_1} + m\mu_{S2} - 1/2\left[(1-m)^2\sigma^2_{S_1} + m^2\sigma^2_{S_2} + 2\rho m(1-m)\sigma_{S_1}\sigma_{S_2}\right]\right) t\right] \\ & \quad \exp\left[-(1-m)\left(\mu_{S_1} - \frac{1}{2}\sigma^2_{S_1}\right) t\right] \exp\left[-m\left(\mu_{S_2} - \frac{1}{2}\sigma^2_{S_2}\right) t\right] \left(\frac{S_{1,t}}{S_{1,0}}\right)^{[1-m]} \left(\frac{S_{2,t}}{S_{2,0}}\right)^{m}. \end{aligned}$$

Therefore, we deduce:

$$\begin{aligned} C_t &= C_0 \left(\frac{S_{1,t}}{S_{1,0}}\right)^{[1-m]} \left(\frac{S_{2,t}}{S_{2,0}}\right)^{m} \times \exp\left[\left(\begin{array}{c} -1/2\left[(1-m)^2\sigma^2_{S_1} + m^2\sigma^2_{S_2} + 2\rho m(1-m)\sigma_{S_1}\sigma_{S_2}\right] \\ +(1-m)\left(\frac{1}{2}\sigma^2_{S_1}\right) + m\left(\frac{1}{2}\sigma^2_{S_2}\right) \end{array}\right) t\right], \\ &= \alpha_t . S_{1,t}^{(1-m)} S_{2,t}^{m} = \alpha_t . S_{1,t} \left(\frac{S_{2,t}}{S_{1,t}}\right)^{m}, \end{aligned}$$

where

$$\alpha_t = C_0 \left(\frac{1}{S_{1,0}}\right)^{[1-m]} \left(\frac{1}{S_{2,0}}\right)^{m} \exp\left[\beta t\right],$$

and

$$\begin{aligned} \beta &= -1/2\left[(1-m)^2\sigma^2_{S_1} + m^2\sigma^2_{S_2} + 2\rho m(1-m)\sigma_{S_1}\sigma_{S_2}\right] + (1-m)\left(\frac{1}{2}\sigma^2_{S_1}\right) + m\left(\frac{1}{2}\sigma^2_{S_2}\right), \\ &= -1/2\left[\left[(1-m)^2 - (1-m)\right]\sigma^2_{S_1} + \left[m^2 - m\right]\sigma^2_{S_2} + 2\rho m(1-m)\sigma_{S_1}\sigma_{S_2}\right], \\ &= -1/2 m(m-1)\widetilde{\sigma}^2. \end{aligned}$$

Therefore, the portfolio value is given by:

$$V_t^{CPPP} = \delta S_{1,t} + \alpha_t . S_{1,t}^{(1-m)} S_{2,t}^{m}.$$

Note that we have: $V_t^{CPPP} = S_{1,t} V_t^{CPPI}(t, T, r = 0, S = S_2/S_1)$.

6.2 Equality of expectations

Recall that:

$$V_T^{OBPP} = q\left[aS_{1,T} + V^{exc}\left(T, T, S_2, aS_1\right)\right], \tag{14}$$

with

$$q = \frac{V_0}{aS_{1,0} + V^{exc}\left(0, T, S_2, aS_1\right)},$$

where the value $V^{exc}\left(0, T, S_2, aS_1\right)$ is equal to:

$$V^{exc}\left(0, T, S_2, aS_1\right) = S_1 Call\left(t, T, S_2/S_1, a, r = 0, \widetilde{\sigma}\right). \tag{15}$$

Recall also that $(\delta = qa)$

$$V_T^{CPPP} = qaS_{1,T} + \alpha_T . S_{1,T}^{1-m} S_{2,T}^{m},$$

where

$$\alpha_T = \left(\frac{V_0 - qaS_{1,0}}{S_{1,0}^{1-m} S_{2,0}^m}\right) \exp\left[\beta T\right] \text{ and } \beta = -\frac{1}{2}m(m-1)\widetilde{\sigma}^2.$$

We are looking for the value of the multiple m such that the expected portfolio values are equal:

$$E[qaS_{1,T} + q\left(S_{2,T} - aS_{1,T}\right)^+] = E[qaS_{1,T} + \alpha_T.S_{1,T}^{(1-m)} S_{2,T}^m].$$

We get the equivalent condition:

$$qE\left[\left(S_{2,T} - aS_{1,T}\right)^+\right] = \alpha_T E\left[S_{1,T}^{(1-m)} S_{2,T}^m\right].$$

$$qE\left[\left(S_{2,T} - aS_{1,T}\right)^+\right] = \left(V_0 - qaS_{1,0}\right) \exp\left[\beta T\right] E\left[\frac{S_{1,T}^{(1-m)}}{S_{1,0}^{(1-m)}} \frac{S_{2,T}^m}{S_{2,0}^m}\right].$$

But we have:

(1)

$$E\left[\frac{S_{1,T}^{(1-m)}}{S_{1,0}^{(1-m)}} \frac{S_{2,T}^m}{S_{2,0}^m}\right] = \exp\left[-\beta T + \mu_{S_1} T + m\left(\mu_{S_2} - \mu_{S_1}\right) T\right];$$

(2)

$$\left(V_0 - qaS_{1,0}\right) = qS_1 Call\left(t, T, S_2/S_1, a, r = 0, \widetilde{\sigma}\right);$$

(3)

$$E\left[\left(S_{2,T} - aS_{1,T}\right)^+\right] = \exp\left[\mu_{S_2} T\right] C(0, T, S_2/S_1, a, r = \mu_{S_2} - \mu_{S_1}, \widetilde{\sigma}).$$

Therefore, we deduce:

$$qS_1 \exp\left[\mu_{S_2} T\right] C(0, T, S_2/S_1, a, r = \mu_{S_2} - \mu_{S_1}, \widetilde{\sigma}) = qS_1 Call\left(t, T, S_2/S_1, a, r = 0, \widetilde{\sigma}\right) \exp\left[\mu_{S_1} T + m\left(\mu_{S_2} - \mu\right.\right.$$

Recall that we assume $\mu_{S_1} < \mu_{S_2}$. Finally, we obtain the value of the multiple m as a function $m^*(a)$ of the performance participation a:

$$m^*(a) = 1 + \left(\frac{1}{\left(\mu_{S_2} - \mu_{S_1}\right) T}\right) \ln\left(\frac{C(0, T, S_2/S_1, a, r = \mu_{S_2} - \mu_{S_1}, \widetilde{\sigma})}{C(0, T, S_2/S_1, a, r = 0, \widetilde{\sigma})}\right).$$

6.3 Hessians of the two portfolio values

For the OBPP strategy, the Hessian H (which corresponds to the mathematical determinant of the matrix of derivatives at the second order) is given by:

$$\begin{aligned} H^{OBPP} &= \frac{\partial^2 V_t^{OBPP}}{\partial S_{1,t}^2} \times \frac{\partial^2 V_t^{OBPP}}{\partial S_{2,t}^2} - \left(\frac{\partial^2 V_t^{OBPP}}{\partial S_{1,t} \partial S_{2,t}}\right)^2 \\ &= \frac{q^2 N'\left(d_{1,t}\right)}{\widetilde{\sigma}^2 \left(T-t\right) S_{1,t}} \left(a \frac{N'\left(d_{2,t}\right)}{S_{2,t}} - \frac{N'\left(d_{1,t}\right)}{S_{1,t}}\right). \end{aligned}$$

Since we have $\frac{N'(d_{1,t})}{N'(d_{2,t})} = \frac{aS_{1,t}}{S_{2,t}}$, we deduce that the Hessian of the OBPP is null.
For the CPPP, the Hessian H is equal to:

$$\begin{aligned}
H^{CPPP} &= \frac{\partial^2 V_t^{CPPP}}{\partial S_{1,t}^2} \times \frac{\partial^2 V_t^{CPPP}}{\partial S_{2,t}^2} - \left(\frac{\partial^2 V_t^{CPPP}}{\partial S_{1,t} \partial S_{2,t}} \right)^2 \\
&= \alpha_t^2 m^2 \left(m-1\right)^2 \left(S_{1,t}^{-1} S_t^m S_{1,t}^{-1} S_t^{m-2} - S_{1,t}^{-2} S_t^{2m-2} \right) \\
&= \alpha_t^2 m^2 \left(m-1\right)^2 S_{1,t}^{-2} \left(S_t^m S_t^{m-2} - S_t^{2m-2} \right) \\
&= 0
\end{aligned}$$